

Dimension free L^p -bounds of maximal functions associated to products of Euclidean balls

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Abstract. A few years ago, Bourgain proved that the centered Hardy-Littlewood maximal function for the cube has dimension free L^p -bounds for $p > 1$. We extend his result to products of Euclidean balls of different dimensions. In addition, we provide dimension free L^p -bounds for the maximal function associated to products of Euclidean spheres for $p > \frac{N}{N-1}$ and $N \geq 3$, where $N - 1$ is the lowest occurring dimension of a single sphere. The aforementioned result is obtained from the latter one by applying the method of rotations from Stein's pioneering work on the spherical maximal function.

Contents

1	Introduction	1
2	Independence of the number of factors	3
3	Stein's approach revisited	14
4	Higher Fourier derivatives of spherical measures	17

1 Introduction

For any convex body B and any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we denote the centered Hardy-Littlewood maximal function of f associated to B by

$$M_B f(x) := \frac{1}{|B|} \sup_{t>0} \int_B |f(x + ty)| \, dy.$$

The history of dimension free bounds starts with the celebrated result by Stein [9], who discovered that for $1 < p \leq \infty$, the centered Hardy-Littlewood maximal operator

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associated to the Euclidean ball in \mathbb{R}^n has an L^p -bound that does not depend on the dimension n . This result has been obtained by averaging over spheres, and using the L^p -boundedness of the spherical maximal operator for $n > 2$ and $p > \frac{n}{n-1}$ (see also [10]). One can ask whether this holds for any convex body B , and several more results have been obtained since then. Bourgain [1] discovered that the centered maximal operator is L^2 -bounded independently of n for any convex, centrally symmetric body in \mathbb{R}^n . The latter result has been improved to the range $p > 3/2$ in [2], and independently by Carbery [4]. Going further, Müller [7] showed that for $p > 1$, one can bound the norm of the maximal operator on L^p in terms of certain geometric invariants. These estimates allowed in particular to give dimension free bounds on the maximal operators associated to the ℓ^q -balls for $1 \leq q < \infty$. However, the case $q = \infty$, i.e. the case of the maximal operator associated to the unit cube, remained open until recently. In [3], Bourgain succeeded to show that also in this case, there exists a dimension free bound. A survey of all these results, with attention to further details, has recently been published by Deleaval, Guédon, and Maurey [5]. Following the latest result by Bougain, the purpose of this work is to prove the following theorem.

Theorem 1. *Let $B := B_1 \times \cdots \times B_\ell$ be a direct product of $\ell \geq 1$ Euclidean balls B_k in \mathbb{R}^{n_k} , $n_k \geq 1$, and put $n := n_1 + \dots + n_\ell$. Assume further that $1 < p \leq \infty$. Then*

$$\|M_B f\|_p \leq C_p \|f\|_p \quad (1.1)$$

for every $f \in L^p(\mathbb{R}^n)$, where C_p depends only on p , but not on ℓ or the dimensions n_k of the factors.

An outline of our approach goes as follows. A naïve way would be to estimate

$$\begin{aligned} M_B f(x) &\leq \frac{1}{|B|} \sup_{t_1, \dots, t_\ell > 0} \int_{B_\ell} \cdots \int_{B_1} |f(x + (t_1 y^{(1)}, \dots, t_\ell y^{(\ell)}))| dy^{(1)} \dots dy^{(\ell)} \\ &\leq \frac{1}{|B_\ell|} \sup_{t_\ell > 0} \int_{B_\ell} \cdots \frac{1}{|B_1|} \sup_{t_1 > 0} \int_{B_1} |f(x + (t_1 y^{(1)}, \dots, t_\ell y^{(\ell)}))| dy^{(1)} \dots dy^{(\ell)}, \end{aligned}$$

getting iterated maximal functions, where we write $x = (x^{(1)}, \dots, x^{(\ell)})$ with $x^{(k)} \in \mathbb{R}^{n_k}$. Due to Stein's dimension free bound for the Euclidean ball, we can estimate each iterated maximal function to get

$$\|M_B f\|_p \leq C_p^\ell \|f\|_p \quad (1.2)$$

for $1 < p < \infty$. Let $B^{(N)}$ be the Euclidean ball in \mathbb{R}^N with Lebesgue measure 1. Since the Fourier transform of $\chi_{B^{(N)}}$ is $\mathcal{O}(|\xi|^{-\frac{N+1}{2}})$ as $|\xi| \rightarrow \infty$, we see that the Fourier transform of χ_B has a decay of at least this rate (with $N = \min_{1 \leq k \leq \ell} n_k$) on certain subspaces of \mathbb{R}^n , while the decay is even better elsewhere, behaving in a way very similar to the cube. Hence, in order to show Theorem 1, we shall make use of some of the central arguments in Bourgain's approach for the cube to attain a bound that depends on p and $\max_{1 \leq k \leq \ell} n_k$, but not on ℓ . From here on, we aim to combine this with (1.2) to achieve a bound as in

Theorem 1. For this, we will provide a similar theorem for spheres. Let $S := S_1 \times \cdots \times S_\ell$ be a product of Euclidean spheres S_k in \mathbb{R}^{n_k} . Let σ_S be the product of the spherical measures of the S_k and define the maximal operator M_S by

$$M_S f(x) := \frac{1}{|S|} \sup_{t>0} \int_S |f(x + t\omega)| d\sigma_S(\omega).$$

Here, $|S|$ denotes the $(n - \ell)$ -dimensional volume of S .

Theorem 2. *Let S be as above and $n_k \geq 3$ for each k . Put $N = \min_{1 \leq k \leq \ell} n_k$ and assume that $p > \frac{N}{N-1}$. Then we have*

$$\|M_S f\|_p \leq C_p \|f\|_p \quad (1.3)$$

for every $f \in L^p(\mathbb{R}^n)$, where C_p only depends on p .

As in the well-known case of $\ell = 1$, the lower bound for p in Theorem 2 is optimal. We will prove Theorem 2 by using Stein's approach to see that we can increase the dimension of each factor of S without increasing C_p . With Stein's argument, we also get $\|M_B f\|_p \leq \|M_S f\|_p$ if B is the convex hull of S , hence Theorem 2 is sufficient for N large enough, depending on a fixed value of p . To show Theorem 2, we proceed with applying an interpolation similar as in Carbery's proof for $p > 3/2$, in which he makes use of the fact that $\langle \xi, \nabla \widehat{\chi_B}(\xi) \rangle$ has bounded L^2 -multiplier norm for a general convex body B . In his final remark, he states that if this derivative has bounded L^q -multiplier norm for a bigger range of q , we would obtain a better bound on p than $3/2$, which is the case if we can bound the higher fractional derivatives

$$\left(\frac{d}{dr} \right)^z \widehat{\sigma_{S^{N-1}}}(r\xi)$$

of the Fourier transforms of spherical measures, where $\text{Re}(z) = \frac{N+1}{2}$. Bounding these uses several ideas from Müller's proof, where he proceeded similarly for arbitrarily higher derivatives to bound them in terms of certain geometric invariants. Also, a lot of the calculations will rely on the explicit forms and the decay of $\widehat{\sigma_{S^{N-1}}}$ and $\widehat{\sigma_S}$.

Hence, if we fix $p > 1$, we have attained a dimension free bound for N large enough by generalizing Stein's and Carbery's ideas. The remaining cases are covered by our generalization of Bourgain's arguments for the cube, achieving a bound only depending on the finitely many remaining N and thus only on p .

2 Independence of the number of factors

This part is mainly a walkthrough of [3], where we will omit any proof that does not need any further modification.

Let B , ℓ , and n be as in Theorem 1 and let $m(\xi) = \widehat{\chi_B}(\xi)$. We group the variables by setting

$$V_k := \left\{ \sum_{j=1}^{k-1} n_j + 1, \dots, \sum_{j=1}^k n_j \right\}, \quad k \in \{1, \dots, \ell\}.$$

The goal of this section is to show the following weaker result.

Proposition 2.1. *Let $N := \max_{1 \leq k \leq \ell} n_k$. Then*

$$\|M_B f\|_p \leq C_{p,N} \|f\|_p \quad (2.1)$$

for every $f \in L^p(\mathbb{R}^n)$, $1 < p \leq \infty$.

It is enough to consider the case $B = (B^{(N)})^\ell$: Since $\|M_B\|_{p \rightarrow p}$ is invariant under linear transformations of B (as already mentioned in [1]), we can assume that $|B_k| = 1$ for every $k \in \{1, \dots, \ell\}$ (hence $B_k = B^{(n_k)}$). By change of coordinates, we can assume $B = \prod_{j=1}^N (B^{(j)})^{\ell_j}$ for certain $\ell_j \in \mathbb{N}$, where, without loss of generality, we allow that $\ell_j = 0$. Suppose that we already found constants $C_{p,j}$ independent of ℓ_j such that

$$\|M_{(B^{(j)})^{\ell_j}} f\|_p \leq C_{p,j} \|f\|_p.$$

Then we can argue similarly as in (1.2) to get

$$\|M_B f\|_p \leq \prod_{j=1}^{\ell} C_{p,j} \|f\|_p.$$

For $B = (B^{(N)})^\ell$, we first consider the same decomposition as in Bourgain's proof. Let $H : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\hat{H}(\xi) = e^{-|\xi|^2}$ and put $\Omega^{(s)} = \chi_B * (H_{2^s} - H_{2^{-s+1}})$ for $s \geq 1$. Then

$$\chi_B = (\chi_B * H) + \sum_{s=1}^{\infty} \Omega^{(s)}.$$

B is in isotropic position, with

$$\begin{aligned} L(B)^2 &= \int_B |\langle x, \xi \rangle|^2 dx = \sum_{k=1}^{\ell} \int_B |\langle x^{(k)}, \xi^{(k)} \rangle|^2 dx + \sum_{k,k'=1}^{\ell} \int_B \langle x^{(k)}, \xi^{(k)} \rangle \langle x^{(k')}, \xi^{(k')} \rangle dx \\ &= \sum_{k=1}^{\ell} \int_{B^{(N)}} |\langle y, \xi^{(k)} \rangle|^2 dy \\ &= L(B^{(N)})^2 \end{aligned}$$

for every $\xi \in S^{n-1}$, since $|B^{(N)}| = 1$. Using only the well-known estimates $|\xi| |m(\xi)| < CL(B)^{-1} < C'$ and $|\langle \xi, \nabla m(\xi) \rangle| < C$ for general convex bodies, Lemma 3 from [1] and the exponential decay of H give us (see (1.16) in [3])

$$\left\| \sup_{t>0} |f * (\Omega^{(s)})_t| \right\|_2 < C 2^{-s/2} \|f\|_2$$

for $s \geq 1$ and

$$\left\| \sup_{t>0} |f * (H * \chi_B)_t| \right\|_2 < C \|f\|_2.$$

For $1 < p < 2$, it suffices to find a bound

$$\left\| \sup_{t>0} |f * (\chi_B * H_{2^{-s}})_t| \right\|_p \leq C_{p,s} \|f\|_p,$$

$s \in \mathbb{N}$, so that $C_{p,s}$ is suitable for interpolation with the L^2 -estimates. For this, Bourgain takes the ideas from [7] to conclude that it suffices to find an L^p -bound for the operator T , defined by

$$\widehat{Tf}(\xi) = |\xi| m(\xi) e^{-4^{-s}|\xi|^2} \hat{f}(\xi), \quad (2.2)$$

and to estimate

$$\sup_{|\xi|=1} \int_{\mathbb{R}^n} |\langle x, \xi \rangle|^k (\chi_B * H_{2^{-s}})(x) dx < C_k, \quad k \geq 1. \quad (2.3)$$

For (2.3), we can make use of the fact that B is symmetric in each coordinate, applying Khinchin's inequality as in [3, p. 279].

To estimate the operator T in (2.2), Bourgain uses a duality argument as in [7] and Stein's dimension free bound on the Riesz transforms (see [9]), which leaves him with proving Lemma 3 from [3]. In our situation, Proposition 2.1 follows from the following Lemma.

Lemma 2.2. *Let $N := \max_{1 \leq k \leq \ell} n_k$. For $R \geq 2$ and $j \in \{1, \dots, n\}$, let $\mu_j = \partial_j(\chi_B * H_{1/R})$. Then for every $2 \leq p < \infty$, $0 < \varepsilon < 1$, and $f \in L^p$ we have*

$$\left\| \left(\sum_{j=1}^n |f * \mu_j|^2 \right)^{1/2} \right\|_p \leq C_{p,\varepsilon,N} R^{12N \cdot \varepsilon} \|f\|_p, \quad (2.4)$$

with $C_{p,\varepsilon,N}$ independent of R and ℓ .

Fix $2 \leq p < \infty$, $R \geq 2$, and $0 < \varepsilon < 1$. The direct interpolation from [3, p. 280] shows that

$$\left\| \left(\sum_{j=1}^n |f * \mu_j|^2 \right)^{1/2} \right\|_p \leq C_p R^{1-2/p} \|f\|_p. \quad (2.5)$$

We proceed with Bourgain's Fourier localization. By means of Pisier's result [8, p. 390], we get the following.

Lemma 2.3. *Let $\eta = (1 - |x|)_+$, $t > 0$ and let $T_j: L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ be the convolution by η_t in the j -th variable. For every $k \in \{1, \dots, \ell\}$ let*

$$S_k := \prod_{j \in V_k} T_j.$$

Furthermore, put for every $k \in \{0, \dots, \ell\}$

$$A_k := \sum_{\substack{A \subset \{1, \dots, \ell\} \\ |A|=k}} \prod_{j \notin A} S_j \prod_{j \in A} (\text{Id} - S_j). \quad (2.6)$$

Then for $1 < q < \infty$, $\|A_k\|_{q \rightarrow q} \leq C_q^k$ with C_q independent of k .

Fix $t := R^{-\varepsilon}$ and let A_k be as in (2.6). Then $A_k \geq 0$ and $\sum_{k=0}^{\ell} A_k = \text{Id}$. For some $K \geq 1$ to be chosen later, we decompose $f \in L^p$ as

$$f = \sum_{k=0}^K A_k f + g.$$

To achieve a good L^2 -estimate on $\left(\sum_{j=1}^n |g * \mu_j|^2\right)^{1/2}$, we poof a variant of Lemma 6 in [3], where we write $\xi = (\zeta_1, \dots, \zeta_\ell)$ with each $\zeta_k \in \mathbb{R}^N$.

Lemma 2.4. *For every $\delta > 0$ and $k \geq 1$, we have*

$$|m(\xi)| \leq C_{k,N} \left(1 + \sum_{|\zeta_j| \leq R^\delta} |\zeta_j|^2\right)^{-k/2} R^{\delta k}. \quad (2.7)$$

Proof. To adapt the original proof to our setting, we need to show

$$|\widehat{\chi_{B^{(N)}}}(\zeta)| \leq e^{-c|\zeta|^2} \quad (2.8)$$

for $|\zeta| \leq 1$ and

$$|\widehat{\chi_{B^{(N)}}}(\zeta)| \leq C \quad (2.9)$$

for $|\zeta| > 1$, with $C < 1$, $\zeta \in R^N$. Since

$$\widehat{\chi_{B^{(N)}}}(\zeta) = r_N^{N/2} |\zeta|^{-N/2} J_{\frac{N}{2}}(2\pi r_N |\zeta|)$$

with J_ν being the Bessel function of order ν and r_N being the radius of $B^{(N)}$, i.e. $r_N = \pi^{-1/2} \Gamma(\frac{N}{2} + 1)^{1/N}$, we use the well-known series expansion

$$J_\nu(x) = \pi^{-1/2} \frac{x^\nu}{2^\nu} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k + \frac{1}{2})}{\Gamma(k + \nu + 1)} \frac{x^{2k}}{(2k)!} \quad (2.10)$$

to get

$$\widehat{\chi_{B^{(N)}}}(\zeta) = r_N^N \pi^{\frac{N-1}{2}} \sum_{k=0}^{\infty} (-1)^k \frac{(2\pi r_N |\zeta|)^{2k}}{(2k)!}.$$

From this series expansion, (2.8) follows. For (2.9), we use that

$$\widehat{\chi_{B^{(N)}}}(\zeta) = \frac{\Gamma(\frac{N}{2} + 1)}{\Gamma(\frac{N+1}{2}) \pi^{1/2}} \int_{-1}^1 e^{it \cdot 2\pi r_N |\zeta|} (1 - t^2)^{\frac{N-1}{2}} dt. \quad (2.11)$$

An application of the van der Corput lemma to the integral in (2.11) yields

$$\begin{aligned}
\left| \int_{-1}^1 e^{it \cdot 2\pi r_N |\zeta|} (1-t^2)^{\frac{N-1}{2}} dt \right| &\leq (2\pi r_N |\zeta|)^{-1} \int_{-1}^1 \left| \frac{d}{dt} (1-t^2)^{\frac{N-1}{2}} \right| dt \\
&= (2\pi r_N |\zeta|)^{-1} (N-1) \cdot 2 \int_0^1 t (1-t^2)^{\frac{N-3}{2}} dt \\
&= (2\pi r_N |\zeta|)^{-1} (N-1) \frac{\Gamma(\frac{N-1}{2})}{\Gamma(\frac{N+1}{2})} \\
&= (\pi r_N |\zeta|)^{-1}.
\end{aligned}$$

Thus

$$|\widehat{\chi_{B^{(N)}}}(\zeta)| \leq (\pi r_N |\zeta|)^{-1} \frac{\Gamma(\frac{N}{2} + 1)}{\Gamma(\frac{N+1}{2}) \pi^{1/2}} = \frac{1}{\pi |\zeta|} \frac{\Gamma(\frac{N}{2} + 1)^{1-1/N}}{\Gamma(\frac{N+1}{2}) \pi^{1/2}},$$

and by Stirling's formula, we get

$$\frac{\Gamma(\frac{N}{2} + 1)^{1-1/N}}{\Gamma(\frac{N+1}{2}) \pi^{1/2}} = |B_{r_N}^{(N-1)}| < \sqrt{e}.$$

Hence for $|\zeta| > 1$, we have

$$|\widehat{\chi_{B^{(N)}}}(\zeta)| \leq \frac{\sqrt{e}}{\pi},$$

allowing us to conclude the Lemma as in [3].

q.e.d.

With Lemma 2.4 we can establish a bound

$$\left\| \left(\sum_{j=1}^n |g * \mu_j|^2 \right)^{1/2} \right\|_2 \leq C_K R^{1-\frac{\varepsilon K}{10}} \|f\|_2, \quad (2.12)$$

with C_K only depending on K . To achieve (2.12), one simply has to replace ξ_j by ζ_j and $\hat{\eta}(t\xi_j)$ by $\hat{\eta}(t\zeta_{j,1}) \cdots \hat{\eta}(t\zeta_{j,N})$ in the corresponding proofs from [3]. By interpolation with (2.5), the choice $K = \lceil \frac{10(p-1)}{\varepsilon} \rceil$ gives us

$$\left\| \left(\sum_{j=1}^n |g * \mu_j|^2 \right)^{1/2} \right\|_p \leq C_{p,\varepsilon} \|f\|_p.$$

Hence it only remains to estimate

$$\left\| \left(\sum_{j=1}^n |A_k f * \mu_j|^2 \right)^{1/2} \right\|_p \leq C_{p,\varepsilon} R^{12N \cdot \varepsilon} \|f\|_p \quad (2.13)$$

for $0 \leq k \leq K$. For $S \subset \{1, \dots, n\}$, put

$$\Gamma_S := \prod_{j \notin S} S_j \prod_{j \in S} (\text{Id} - S_j).$$

Then, by the triangle inequality, we have

$$\left(\sum_{j=1}^n |A_k f * \mu_j|^2 \right)^{1/2} \leq \left(\sum_{j=1}^n \left| \sum_{\substack{|S|=k \\ j \notin S}} \Gamma_S f * \mu_j \right|^2 \right)^{1/2} \quad (2.14)$$

$$+ \left(\sum_{j=1}^n \left| \sum_{\substack{|S|=k \\ j \in S}} \Gamma_S f * \mu_j \right|^2 \right)^{1/2}. \quad (2.15)$$

Bourgain applies a stochastic method to decouple the variables, which reduces (2.14) to the case $k = 0$ and (2.15) to the case $k = 1$. We can acquire the same by replacing T_j by S_j in that procedure. With that, we only have to find suitable constants $b_0 = b_0(R)$, $b_1 = b_1(R)$ such that

$$\left\| \left(\sum_{j=1}^n |A_0 f * \mu_j|^2 \right)^{1/2} \right\|_p \leq b_0 \|f\|_p \quad (2.16)$$

and

$$\left\| \left(\sum_{k=1}^\ell |\Gamma_k G_k f|^2 \right)^{1/2} \right\|_p \leq b_1 \|f\|_p, \quad (2.17)$$

where $\Gamma_k = (\text{Id} - S_k) \prod_{j \neq k} S_j$ and

$$G_k f = \left(\sum_{j \in V_k} |f * \mu_j|^2 \right)^{1/2}.$$

Let $B_p = B_{p,R}$ be minimal such that

$$\left\| \left(\sum_{j=1}^n |f * \mu_j|^2 \right)^{1/2} \right\|_p \leq B_p \|f\|_p. \quad (2.18)$$

To estimate b_1 , we need to rely on Lemmas 7-9 in [3]. The proofs of these will become more complicated in our setting, with estimates that will depend on N . For the proofs, we will provide slightly more details than in [3]. Instead of using the properties of the convolution operators T_j , we need to convolve with a function that is roughly stable under small translations. Bourgain considers the function

$$\varphi(x) := \frac{c}{1 + x^4}$$

with c so that $\int_{\mathbb{R}} \varphi(x) dx = 1$. Then $C^{-1}\varphi(x - y) \leq \varphi(x) \leq C\varphi(x - y)$ for every $x \in \mathbb{R}$ and $|y| \leq 1$, $\varphi(x)$ is $\mathcal{O}(e^{-C|x|})$ as $|x| \rightarrow \infty$, and $|1 - \hat{\varphi}(x)| < Cx^2$ for every $x \in \mathbb{R}$. Fix $t_0 := R^{-3\epsilon}$, and let $\tilde{L}_j: L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ be the convolution by φ_{t_0} in the j -th variable, $j \in \{1, \dots, n\}$. For $k \in \{1, \dots, \ell\}$, let

$$L_k := \prod_{j \in V_k} \tilde{L}_j \quad \text{and} \quad L^{(k)} := \prod_{j \notin V_k} \tilde{L}_j = \prod_{k' \neq k} L_{k'}.$$

With these notions, we show the following version of Lemma 7 from [3].

Lemma 2.5. Let $M \in \mathbb{N}$, $q = 2^M$, and let $f_1, \dots, f_\ell \in L^q(\mathbb{R}^n)$ be positive functions. Then

$$\left\| \sum_{j=1}^{\ell} L^{(j)} f_j \right\|_q \leq C_{q,N} \sum_{k=0}^{M-1} \left\| \left(\prod_{j=1}^{\ell} L_j \right) \left(\sum_{j=1}^{\ell} f_j^{2^k} \right) \right\|_{2^{M-k}}^{2^{-k}} + C_{q,N} \left(\sum_{j=1}^{\ell} \|f_j\|_q^q \right)^{1/q} \quad (2.19)$$

$$\leq C_{q,N} \left\| \sum_{j=1}^{\ell} f_j \right\|_q. \quad (2.20)$$

Proof. The proof of (2.20) is easy. We will show (2.19) by induction on M , with the case $M = 0$ ($q = 1$) being obvious. Fix $M > 0$ and assume that

$$\left\| \sum_{j=1}^n L^{(j)} f_j \right\|_{q/2} \leq C_q \sum_{k=0}^{M-2} \left\| \left(\prod_{j=1}^n L_j \right) \left(\sum_{j=1}^n f_j^{2^k} \right) \right\|_{2^{M-1-k}}^{2^{-k}} + C_q \left(\sum_{j=1}^n \|f_j\|_{q/2}^{q/2} \right)^{2/q}.$$

Then we have

$$\left\| \sum_{j=1}^n L^{(j)} f_j \right\|_q^q \leq q! \sum_{1 \leq j_1 \leq \dots \leq j_q \leq \ell} \int_{\mathbb{R}^n} \prod_{k=1}^q (L^{(j_k)} f_{j_k}(x)) \, dx$$

If we split

$$\begin{aligned} \sum_{1 \leq j_1 \leq \dots \leq j_q \leq \ell} \int_{\mathbb{R}^n} \prod_{k=1}^q (L^{(j_k)} f_{j_k}(x)) \, dx &= \sum_{\substack{j_1 \leq \dots \leq j_q \\ j_1 = j_2}} \int_{\mathbb{R}^n} \prod_{k=1}^q (L^{(j_k)} f_{j_k}(x)) \, dx \\ &\quad + \sum_{\substack{j_1 \leq \dots \leq j_q \\ j_1 < j_2}} \int_{\mathbb{R}^n} \prod_{k=1}^q (L^{(j_k)} f_{j_k}(x)) \, dx, \end{aligned}$$

we can estimate the first sum as follows, using Hölder's inequality with $\frac{q}{2}$ and $\frac{q}{q-2}$

$$\begin{aligned} \sum_{\substack{j_1 \leq \dots \leq j_q \\ j_1 = j_2}} \int_{\mathbb{R}^n} \prod_{k=1}^q (L^{(j_k)} f_{j_k}(x)) \, dx &\leq \int_{\mathbb{R}^n} \left(\sum_{j=1}^n (L^{(j)} f_j(x))^2 \right) \cdot \left(\sum_{j=1}^n L^{(j)} f_j(x) \right)^{q-2} \, dx \\ &\leq \left\| \sum_{j=1}^n (L^{(j)} f_j)^2 \right\|_{q/2} \left\| \sum_{j=1}^n L^{(j)} f_j \right\|_q^{q-2}. \end{aligned} \quad (2.21)$$

In the case $j_1 < j_2$, we estimate

$$\int_{\mathbb{R}^n} \prod_{k=1}^q (L^{(j_k)} f_{j_k}(x)) \, dx$$

directly. Without loss of generality, assume $j_1 = 1$ and put

$$g_j = \left(\prod_{\substack{1 \leq k \leq \ell \\ k \notin \{1, j\}}} L_k \right) f_j$$

for $j \in \{1, \dots, \ell\}$. Denote $x = (x^{(1)}, x')$ with $x^{(1)} \in \mathbb{R}^N$ and $x' \in \mathbb{R}^{n-N}$, and let $\Phi: \mathbb{R}^N \rightarrow \mathbb{R}$, $\Phi(y) = \prod_{k=1}^N \varphi(y_k)$. Then

$$\Phi_{t_0}(y) = \prod_{k=1}^N \varphi_{t_0}(y_k),$$

and we get

$$\begin{aligned} & \int_{\mathbb{R}^n} \prod_{k=1}^q (L^{(j_k)} f_{j_k}(x)) \, dx \\ &= \int_{\mathbb{R}^{n-N}} \int_{\mathbb{R}^N} g_1(x^{(1)}, x') \cdot \prod_{k=2}^q L_1 g_{j_k}(x^{(1)}, x') \, dx^{(1)} \, dx' \\ &= \int_{\mathbb{R}^{n-N}} \int_{(\mathbb{R}^N)^{q-1}} \int_{\mathbb{R}^N} g_1(x^{(1)}, x') \cdot \prod_{k=2}^q (g_{j_k}(x^{(1)} - y^{(k)}, x') \Phi_{t_0}(y^{(k)})) \, dx^{(1)} \, d(y^{(2)}, \dots, y^{(q)}) \, dx'. \end{aligned}$$

Now fix x' . Using that $\varphi_{t_0}(\tau) \geq \varphi_{t_0}(t_0) = \frac{c}{2t_0}$, averaging over $|\tau| \leq t_0$ implies

$$\begin{aligned} & \int_{(\mathbb{R}^N)^{q-1}} \int_{\mathbb{R}^N} g_1(x^{(1)}, x') \cdot \prod_{k=2}^q (g_{j_k}(x^{(1)} - y^{(k)}, x') \Phi_{t_0}(y^{(k)})) \, dx^{(1)} \, d(y^{(2)}, \dots, y^{(q)}) \\ & \leq c^{-N} \int_{[-t_0, t_0]^N} \Phi_{t_0}(\tau) \int_{(\mathbb{R}^N)^{q-1}} \int_{\mathbb{R}^N} g_1(x^{(1)}, x') \\ & \quad \cdot \prod_{k=2}^q (g_{j_k}(x^{(1)} - y^{(k)}, x') \Phi_{t_0}(y^{(k)})) \, dx^{(1)} \, d(y^{(2)}, \dots, y^{(q)}) \, d\tau \\ & = c^{-N} \int_{[-t_0, t_0]^N} \Phi_{t_0}(\tau) \int_{(\mathbb{R}^N)^{q-1}} \int_{\mathbb{R}^N} g_1(x^{(1)} - \tau, x') \\ & \quad \cdot \prod_{k=2}^q (g_{j_k}(x^{(1)} - y^{(k)}, x') \Phi_{t_0}(y^{(k)} + \tau)) \, dx^{(1)} \, d(y^{(2)}, \dots, y^{(q)}) \, d\tau \\ & \leq C^{N(q-1)} c^{-N} \int_{[-t_0, t_0]^N} \Phi_{t_0}(\tau) \int_{(\mathbb{R}^N)^{q-1}} \int_{\mathbb{R}^N} g_1(x^{(1)} - \tau, x') \\ & \quad \cdot \prod_{k=2}^q (g_{j_k}(x^{(1)} - y^{(k)}, x') \Phi_{t_0}(y^{(k)})) \, dx^{(1)} \, d(y^{(2)}, \dots, y^{(q)}) \, d\tau \\ & \leq C_{q,N} \int_{\mathbb{R}^N} \prod_{k=1}^q (L_1 g_{j_k}(x^{(1)}, x')) \, dx^{(1)}. \end{aligned}$$

Altogether, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \prod_{k=1}^q (L^{(j_k)} f_{j_k}(x)) \, dx &\leq C_{q,N} \int_{\mathbb{R}^n} \prod_{k=1}^q (L_1 g_{j_k}(x)) \, dx \\ &= C_{q,N} \int_{\mathbb{R}^n} \left(\prod_{j=1}^n L_j \right) f_1(x) \cdot \prod_{k=2}^q L^{(j_k)} f_{j_k}(x) \, dx. \end{aligned}$$

The same argument holds if $j_1 \neq 1$, and hence by Hölder's inequality with q and $\frac{q-1}{q}$,

$$\begin{aligned} \sum_{\substack{j_1 \leq \dots \leq j_q \\ j_1 < j_2}} \int_{\mathbb{R}^n} \prod_{k=1}^q (L^{(j_k)} f_{j_k}(x)) \, dx &\leq C_{q,N} \sum_{j=1}^n \sum_{1 \leq j_2, \dots, j_q \leq \ell} \int_{\mathbb{R}^n} \left(\prod_{j=1}^n L_j \right) f_j(x) \cdot \prod_{k=2}^q L^{(j_k)} f_{j_k}(x) \, dx \\ &\leq C_{q,N} \left\| \left(\prod_{j=1}^n L_j \right) \left(\sum_{j=1}^n f_j \right) \right\|_q \left\| \sum_{j=1}^n L^{(j)} f_j \right\|_q^{q-1}. \end{aligned} \quad (2.22)$$

With (2.21) and (2.22), we get

$$\begin{aligned} \left\| \sum_{j=1}^n L^{(j)} f_j \right\|_q^q &\leq C_q \left\| \sum_{j=1}^n (L^{(j)} f_j)^2 \right\|_{q/2} \left\| \sum_{j=1}^n L^{(j)} f_j \right\|_q^{q-2} \\ &\quad + C_q \left\| \left(\prod_{j=1}^n L_j \right) \left(\sum_{j=1}^n f_j \right) \right\|_q \left\| \sum_{j=1}^n L^{(j)} f_j \right\|_q^{q-1}, \end{aligned}$$

Since

$$\left\| \sum_{j=1}^n (L^{(j)} f_j)^2 \right\|_{q/2}^{1/2} \leq \left\| \sum_{j=1}^n L^{(j)} f_j \right\|_q$$

and $(L^{(j)} f_j)^2 \leq L^{(j)} f_j^2$, this leads to

$$\left\| \sum_{j=1}^n L^{(j)} f_j \right\|_q \leq C_q \left\| \sum_{j=1}^n L^{(j)} f_j^2 \right\|_{q/2}^{1/2} + C_q \left\| \left(\prod_{j=1}^n L_j \right) \left(\sum_{j=1}^n f_j \right) \right\|_q,$$

and our induction hypothesis concludes the lemma. **q.e.d.**

We directly show a version of Lemma 9 from [3], which is a corollary of Lemma 8.

Lemma 2.6. *Let $M \geq 1$, $q = 2^M$, and $f_1, \dots, f_\ell \in L^q(\mathbb{R}^n)$. Then*

$$\left\| \left(\sum_{k=1}^{\ell} |L^{(k)} G_k f_k|^2 \right)^{1/2} \right\|_q \leq C_{q,N} R^{12N \cdot \varepsilon} \left\| \left(\sum_{k=1}^{\ell} |f_k|^2 \right)^{1/2} \right\|_q. \quad (2.23)$$

Proof. Since $\mu_j = \partial_j(\chi_B * H_{1/R}) = \partial_j(\chi_B) * H_{1/R}$ by taking distributional derivatives and convolutions, and $H_{1/R}$ is the density function of a probability measure, we can use Jensen's inequality to estimate

$$\begin{aligned} \left\| \left(\sum_{k=1}^{\ell} |L^{(k)} G_k f_k|^2 \right)^{1/2} \right\|_q &\leq \left\| H_{1/R} * \left(\sum_{k=1}^{\ell} |L^{(k)} G'_k f_k|^2 \right)^{1/2} \right\|_q \\ &\leq \left\| \left(\sum_{k=1}^{\ell} L^{(k)} |G'_k f_k|^2 \right)^{1/2} \right\|_q, \end{aligned}$$

where

$$G'_k f_k = \left(\sum_{j \in V_k} |(\partial_j \chi_B) * f_k|^2 \right)^{1/2}.$$

Hence it suffices to show

$$\left\| \left(\sum_{k=1}^{\ell} L^{(k)} |G'_k f_k|^2 \right)^{1/2} \right\|_q \leq C_q R^{24N\varepsilon} \left\| \left(\sum_{k=1}^{\ell} |f_k|^2 \right)^{1/2} \right\|_q. \quad (2.24)$$

Take $\psi \in \mathcal{S}(\mathbb{R}^n)$. We want to establish $\langle \partial_j \chi_B, \psi \rangle$ for every $j \in \{1, \dots, n\}$. First, assume $j = \ell = 1$. Then we have

$$\begin{aligned} -\langle \partial_j \chi_B, \psi \rangle &= \int_{B^{(N)}} \partial_1 \psi(x) \, dx = \int_{B_{r_N}^{(N-1)} - \sqrt{r_N^2 - |x'|^2}}^{\sqrt{r_N^2 - |x'|^2}} \int \partial_1 \psi(x_1, x') \, dx_1 \, dx' \\ &= \int_{B_{r_N}^{(N-1)}} \psi(\sqrt{r_N^2 - |x'|^2}, x') - \psi(-\sqrt{r_N^2 - |x'|^2}, x') \, dx' \\ &= \int_{S_{r_N}^{N-1}} \psi(\omega) \frac{\omega_1}{r_N} \, d\sigma(\omega). \end{aligned}$$

For general j and ℓ , choose the unique $k = k(j)$ with $j \in V_k$. Then

$$-\langle \partial_j \chi_B, \psi \rangle = \int_{B^{\hat{k}}} \int_{S_{r_N}^{N-1}} \psi(x^{(1)}, \dots, \underbrace{\omega}_{k\text{-th}}, \dots, x^{(\ell)}) \frac{\omega_{j-(k-1)N}}{r_N} \, d\sigma(\omega) \, dx', \quad (2.25)$$

where $B^{\hat{k}} = \prod_{k' \neq k} B_{k'}$ and $x' = (x_1, \dots, x_{(k-1)N}, x_{kN+1}, \dots, x_n)$. Now let

$$\tau_j f(x) = \int_{S_{r_N}^{N-1}} f(x^{(1)}, \dots, x^{(k)} + \omega, \dots, x^{(\ell)}) \frac{|\omega_{j-(k-1)N}|}{r_N} \, d\sigma(\omega)$$

Then

$$\|\tau_j\|_{1 \rightarrow 1} = 2|B_{r_N}^{(N-1)}| < 2\sqrt{e}$$

and hence

$$\sum_{j \in V_k} |(\partial_j \chi_B) * f_k|^2 \leq \sum_{j \in V_k} |\chi_{B^{\hat{k}}} * \tau_j |f_k||^2 \leq 2\sqrt{e} \sum_{j \in V_k} \tau_j (|f_k|^2 * \chi_{B^{\hat{k}}}),$$

taking the last convolution only in the variables of $B^{\hat{k}}$. Application of Lemma 2.5 gives us

$$\begin{aligned} \left\| \left(\sum_{k=1}^{\ell} L^{(k)} |G'_k f_k|^2 \right)^{1/2} \right\|_q &\leq 2\sqrt{e} \left\| \sum_{k=1}^{\ell} L^{(k)} \left(\sum_{j \in V_k} \tau_j (|f_j|^2 * \chi_{B^{\hat{k}}}) \right) \right\|_{q/2}^{1/2} \\ &\leq C_{q,N} \sum_{i=1}^{M-1} \left\| \left(\prod_{k=1}^{\ell} L_k \right) \left(\sum_{k=1}^{\ell} \left(\sum_{j \in V_k} \tau_j (|f_k|^2 * \chi_{B^{\hat{k}}}) \right)^{2^{i-1}} \right) \right\|_{2^{M-i}}^{2^{-i}} \\ &\quad + C_{q,N} \left(\sum_{k=1}^{\ell} \left\| \sum_{j \in V_k} \tau_j (|f_k|^2 * \chi_{B^{\hat{k}}}) \right\|_{q/2}^{q/2} \right)^{1/q}. \end{aligned} \quad (2.26)$$

We can easily estimate

$$\left(\sum_{k=1}^{\ell} \left\| \sum_{j \in V_k} \tau_j (|f_k|^2 * \chi_{B^{\hat{k}}}) \right\|_{q/2}^{q/2} \right)^{1/q} \leq (2N\sqrt{e})^{1/2} \cdot \left\| \left(\sum_{k=1}^{\ell} |f_k(x)|^2 \right)^{1/2} \right\|_q. \quad (2.27)$$

Note that from the properties of φ , we can deduce

$$\varphi_{t_0}(x + \rho) = \frac{1}{t_0} \frac{c}{1 + \frac{(x+\rho)^4}{t_0^4}} \leq \frac{1}{t_0} \varphi(x + \rho) \leq \frac{C^{\lceil r_N \rceil}}{t_0} \varphi(x) = \frac{C_N}{t_0^5} \frac{1}{\frac{1+x^4}{t_0^4}} \leq \frac{C_N}{t_0^4} \varphi_{t_0}(x)$$

for $x \in \mathbb{R}$ and $|\rho| \in [-r_N, r_N]$. Hence

$$\Phi_{t_0}(x - y) \leq C_N t_0^{-4N} \Phi_{t_0}(x)$$

for every $x, y \in \mathbb{R}^N$ with $|y| \leq r_N$. Thus for any positive $g \in L^q(\mathbb{R}^N)$, we have

$$\begin{aligned} \left(\Phi_{t_0} * \int_{S_{r_N}^{N-1}} (\tau_{\omega} g) \cdot \frac{|\omega_j|}{r_N} d\sigma(\omega) \right)(x) &\leq 2\sqrt{e} C_N t_0^{-4N} (\Phi_{t_0} * g)(x) \\ &= C_N t_0^{-4N} \int_{\mathbb{R}^N} \int_{B^{(N)}} \Phi_{t_0}(y) dz g(x - y) dy \\ &\leq C_N t_0^{-8N} \int_{\mathbb{R}^N} \int_{B^{(N)}} \Phi_{t_0}(y - z) dz g(x - y) dy \\ &= C_N R^{24N \cdot \varepsilon} \cdot (\chi_{B^{(N)}} * \Phi_{t_0}) * g(x). \end{aligned}$$

Taking $1 \leq i \leq N - 1$, this implies

$$\begin{aligned}
& \left\| \left(\prod_{k=1}^{\ell} L_k \right) \left(\sum_{k=1}^{\ell} \left(\sum_{j \in V_k} \tau_j(|f_k|^2 * \chi_{B^k}) \right)^{2^{i-1}} \right) \right\|_{2^{M-i}}^{2^{-i}} \\
& \leq \left\| \left(\sum_{k=1}^{\ell} C_N N R^{24N \cdot \varepsilon} \cdot |f_k|^2 * \chi_B \right)^{2^{i-1}} \right\|_{2^{M-i}}^{2^{-i}} \\
& \leq C_N R^{12N \cdot \varepsilon} \left\| \left(\sum_{k=1}^{\ell} |f_k|^2 \right)^{1/2} \right\|_q.
\end{aligned} \tag{2.28}$$

The estimates (2.26), (2.27), and (2.28) conclude the proof.

q.e.d.

Now, we can estimate (2.16) and (2.17) by arguing as in [3, Section 4]. Since

$$A_0 = \prod_{k=1}^{\ell} S_k = \prod_{j=1}^n T_j,$$

we can establish

$$b_0 \leq C_p(R^{3\varepsilon} + B_p R^{-\frac{2\varepsilon}{p}})$$

with B_p as in (2.18). For (2.17), we can use Lemmas 2.5 and 2.6 to obtain

$$b_1 \leq C_{p,N}(R^{12N \cdot \varepsilon} + B_p R^{-\frac{2\varepsilon}{p}}).$$

This leads to

$$B_p \leq C_{p,\varepsilon}(1 + b_0 + b_1) < C_{p,\varepsilon,N}(R^{12N \cdot \varepsilon} + B_p R^{-\frac{2\varepsilon}{p}}),$$

giving us

$$B_p \leq C_{p,\varepsilon,N} R^{12N \cdot \varepsilon} \tag{2.29}$$

and thus proving Lemma 2.2 and Proposition 2.1.

3 Stein's approach revisited

Let $B = B_1 \times \cdots \times B_{\ell}$, $n = n_1 + \cdots + n_{\ell}$ be as in Theorem 1. Let $N := \min_{1 \leq k \leq \ell} n_k$. Fix $1 < p \leq \infty$. We show that if $N > \frac{p}{p-1}$, i.e. $p > \frac{N}{N-1}$, we can deduce (1.1) in Theorem 1 from Theorem 2, and that (1.3) from Theorem 2 holds if we show the following theorem.

Theorem 2'. *Let $S := (S_R^{N-1})^{\ell}$, with R so that $|S| = 1$. Then*

$$\|M_S f\|_p \leq C_{p,N} \|f\|_p. \tag{3.1}$$

We can freely change the radii of each sphere because for every $r, s > 0$ and $n > 1$, we have

$$\int_{S_r^{n-1}} f(s\omega) d\sigma_{S_r^{n-1}}(\omega) = \frac{1}{s^{n-1}} \int_{S_{sr}^{n-1}} f(\omega) d\sigma_{S_{sr}^{n-1}}(\omega).$$

Thus, if $S' = S'_1 \times \cdots \times S'_\ell$ is a product of spheres with $\dim S'_k = \dim S_k$ for each k , we have $\|M_{S'}f\|_{p \rightarrow p} = \|M_S f\|_{p \rightarrow p}$.

We can not get a pointwise estimate $M_B f(x) \leq M_S f(x)$ as in the case $\ell = 1$, but we can indeed get an L^p -estimate by a similar argument. Assume that each B_k and each S_k has radius 1 (thus $S_k = S^{n_k-1}$), and let σ_k be the respective surface measure for each S_k .

Lemma 3.1. *We have*

$$\|M_B f\|_p \leq \|M_S f\|_p \quad (3.2)$$

for each $f \in L^p(\mathbb{R}^n)$.

Proof. Using polar coordinates and the fact that $|S_k| = n_k |B_k|$, we can estimate

$$\begin{aligned} M_B f(x) &= \frac{1}{|B|} \sup_{t>0} \int_{[0,1]^\ell} \prod_{k=1}^\ell s_k^{n_k-1} \int_{S_1} \cdots \int_{S_\ell} |f(x + t(s_1\omega_1, \dots, s_\ell\omega_\ell))| d\sigma_\ell(\omega_\ell) \dots d\sigma_1(\omega_1) ds \\ &\leq \int_{[0,1]^\ell} \prod_{k=1}^\ell s_k^{n_k-1} \frac{1}{|B|} \prod_{k=1}^\ell s_k^{-n_k+1} \sup_{t>0} \int_{S_1^{n_1-1}} \cdots \int_{S_\ell^{n_\ell-1}} |f(x + t(\omega_1, \dots, \omega_\ell))| \\ &\quad d\sigma_\ell(\omega_\ell) \dots d\sigma_1(\omega_1) ds \\ &= \int_{[0,1]^\ell} \prod_{k=1}^\ell n_k s_k^{n_k-1} M_{S_1^{n_1-1} \times \dots \times S_\ell^{n_\ell-1}} f(x) ds. \end{aligned}$$

A simple application of Minkowski's integral inequality yields

$$\|M_B f\|_p \leq \int_{[0,1]^\ell} \prod_{k=1}^\ell n_k s_k^{n_k-1} \cdot \|M_S f\|_p ds = \|M_S f\|_p,$$

which is (3.2). **q.e.d.**

We now generalize Stein's method of rotations from [9] for our situation with the following Lemma

Lemma 3.2. *Let $S = S^{n_1-1} \times \cdots \times S^{n_\ell-1}$, $k \in \{1, \dots, \ell\}$, and set*

$$S^+ = \prod_{j=1}^{k-1} S^{n_j-1} \times S^{n_k} \times \prod_{j=k+1}^\ell S^{n_j-1}.$$

Let $1 < p \leq \infty$ and assume that there is a constant $C > 0$ such that $\|M_S f\|_p \leq C\|f\|_p$ for every $f \in L^p(\mathbb{R}^n)$. Then also

$$\|M_{S^+} f\|_p \leq C\|f\|_p \quad (3.3)$$

for every $f \in L^p(\mathbb{R}^n)$.

Proof. We can assume $k = 1$. For any $u \in S^{n_1}$, denote by

$$S_u^{n_1-1} := \{x \in S^{n_1} : x \perp u\}$$

the rotated $(n_1 - 1)$ -dimensional spheres in \mathbb{R}^{n_1+1} , and let σ^u be the surface measure of $S_u^{n_1-1}$ so that $\sigma^u(S_u^{n_1-1}) = |S^{n_1-1}|$. Furthermore, let σ_1^+ be the surface measure of S^{n_1} . Define a new measure μ on S^{n_1} by putting for every Lebesgue-measurable set $A \subset S^{n_1}$

$$\mu(A) := \int_{S^{n_1}} \int_{S_u^{n_1-1}} \chi_A(\omega) d\sigma^u(\omega) d\sigma_1^+(u).$$

By [9], we have $\mu = |S^{n_1-1}| \cdot \sigma_1^+$, and

$$\|M_{S_u^{n_1-1}} f\|_p = \|M_{S^{n_1-1}} f\|_p \leq C\|f\|_p.$$

Hence we can calculate

$$\begin{aligned} M_{S^+} f(x) &= \frac{1}{|S^+|} \sup_{t>0} \frac{1}{|S^{n_1-1}|} \int_{S^{n_1}} \int_{S_u^{n_1-1}} \int_{S_2} \cdots \int_{S_\ell} |f(x + t(\omega_1, \dots, \omega_\ell))| \\ &\quad d\sigma_\ell(\omega_\ell) \dots d\sigma_2(\omega_2) d\sigma_1^u(\omega_1) d\sigma_1^+(u) \\ &\leq \frac{1}{|S^{n_1}|} \int_{S^{n_1}} \frac{1}{|S|} \sup_{t>0} \int_{S_u^{n_1-1}} \int_{S_2} \cdots \int_{S_\ell} |f(x + t(\omega_1, \dots, \omega_\ell))| \\ &\quad d\sigma_\ell(\omega_\ell) \dots d\sigma_2(\omega_2) d\sigma_1^u(\omega_1) d\sigma_1^+(u) \\ &= \frac{1}{|S^{n_1}|} \int_{S^{n_1}} M_{S_u} f(x) d\sigma_1^+(u), \end{aligned}$$

where $S_u = S_u^{n_1-1} \times S_2 \times \cdots \times S_\ell$. By Minkowski's integral inequality, we then get

$$\|M_{S^+} f\|_p \leq \frac{1}{|S^{n_1}|} \int_{S^{n_1}} \|M_{S_u} f\|_p d\sigma_1^+(u) \leq C\|f\|_p.$$

This concludes the lemma. **q.e.d.**

Now assume that we've already shown Theorem 2' and take B as in Theorem 1. Fix $1 < p < \infty$ and let $N_0 = \lceil \frac{p}{p-1} \rceil$. By change of coordinates, we can assume $B = B' \times B''$, where

$$B' = \prod_{n_k \geq N_0} B_k \quad \text{and} \quad B'' = \prod_{n_k < N_0} B_k.$$

Then

$$\|M_B\|_{p \rightarrow p} \leq \|M_{B'}\|_{p \rightarrow p} \cdot \|M_{B''}\|_{p \rightarrow p}.$$

Furthermore, assume that for $n_k \geq N_0$, each B_k has radius 1, while for $n_k < N_0$, each B_k has volume 1. Since N_0 only depends on p , we get

$$\|M_{B''}\|_{p \rightarrow p} \leq C_p$$

from Proposition 2.1. Let $S' = \prod_{n_k \geq N_0} S^{n_k-1}$. Then by Lemma 3.1 and successive application of Lemma 3.2, we obtain

$$\|M_{B'}\|_{p \rightarrow p} \leq \|M_{S'}\|_{p \rightarrow p} \leq \|M_{(S^{N_0-1})^{\ell'}}\|_{p \rightarrow p},$$

where $\ell' = |\{k \in \{1, \dots, \ell\} : n_k \geq N_0\}|$. But since we can freely vary the radius of each sphere, Theorem 2' implies

$$\|M_{(S^{N_0-1})^{\ell'}}\|_{p \rightarrow p} \leq C_{N_0} = C_p.$$

This concludes Theorem 1.

4 Higher Fourier derivatives of spherical measures

We are left with proving Theorem 2'. Let $N > 2$, $p > \frac{N}{N-1}$, and $S = (S_R^{N-1})^\ell$ with R so that $|S| = 1$, i.e. $R^{N-1} = \frac{\Gamma(N/2)}{2\pi^{N/2}}$.

First, we use the approach from [4] to show that we only have to bound

$$\left\| \sup_{1 \leq t \leq 2} \int_S |f(x + t\omega)| d\sigma_S(\omega) \right\|_p \leq C_{p,N} \|f\|_p \quad (4.1)$$

for $p < 2$. For our setting, we use a different proof, which can be found in Lemma 6.15 and the argument in subsection 6.5.1 from [5]. This result makes use of Lemma 3 from [1]. Since the proofs only rely on the properties of the respective Fourier transforms, a short inspection of them shows that these lemmas still hold when we take finite (signed) Borel measures on \mathbb{R}^n instead of L^1 -kernels, in the following sense.

Lemma 4.1 (Lemma 3 from [1]). *Let ν be a finite Borel measure on \mathbb{R}^n so that $\hat{\nu}$ is differentiable, and put*

$$\alpha_j := \max_{2^j \leq |\xi| \leq 2^{j+2}} |\hat{\nu}(\xi)|, \quad \beta_j := \max_{2^j \leq |\xi| \leq 2^{j+2}} |\langle \nabla \hat{\nu}(\xi), \xi \rangle|$$

for every $j \in \mathbb{Z}$. Then for every $f \in L^2$, we have

$$\left\| \sup_{t>0} |f * \nu_t| \right\|_2 \leq C\Gamma(\nu) \|f\|_2$$

with

$$\Gamma(\nu) := \sum_{j \in \mathbb{Z}} \alpha_j^{1/2} (\alpha_j^{1/2} + \beta_j^{1/2}). \quad (4.2)$$

Lemma 4.2 (Lemma 6.15 from [5]). *Let ν be a finite Borel measure on \mathbb{R}^n and $K \in L^1(\mathbb{R}^n)$ such that $\hat{\nu}$ and \hat{K} are both differentiable. Assume there is a constant C so that for every $\theta \in S^{n-1}$ and every $u \in \mathbb{R}^*$*

$$|\hat{\mu}(u\theta)| \leq C \cdot \min\{|u|, |u|^{-1}\}, \quad (4.3)$$

$$|\langle \theta, \nabla \hat{\mu}(u\theta) \rangle| \leq C \cdot \min\{1, |u|^{-1}\} \quad (4.4)$$

with $\mu = \nu, K$. Then we have

$$\Gamma(\nu * K_{2^k}) \leq C' 2^{-|k|/2} \quad (4.5)$$

for every $k \in \mathbb{Z}$, with C' only depending on C and $\Gamma(\nu * K_{2^k})$ as in (4.2).

Let P be the Poisson kernel, i.e. $\hat{P}(\xi) = e^{-|\xi|}$. We will show later that the Borel measure $\sigma_S - P \, dx$ satisfies (4.3) and (4.4). By Stein's maximal theorem for semigroups $\|\sup_{t>0} P_t * f\|_q < C\|f\|_q$ for $1 < q < \infty$, and one can take $\hat{K}(\xi) = e^{-|\xi|} - e^{-2|\xi|}$ in Lemma 4.2 to proceed as in [5], getting the strong L^2 -boundedness property from Carbery's proof as required, and being left with showing (4.1). Here, we use part (ii) of the proposition in [4], which also holds for any finite Borel measure with bounded Fourier transform. For this, we need to consider fractional derivatives. For a finite Borel measure ν on \mathbb{R}^n and $z \in \mathbb{C}$, denote the fractional derivative of ν of order α by

$$(\langle \xi, \nabla \rangle)^z \hat{\nu}(\xi) = \left(\frac{d}{dr} \right)^z \hat{\nu}(r\xi) \Big|_{r=1} = \int (2\pi i \langle x, \xi \rangle)^z e^{2\pi i \langle \cdot, \xi \rangle} d\nu(x) \quad (4.6)$$

whenever the right hand side is well-defined. Let

$$m(\xi) = \widehat{\sigma_S}(\xi).$$

According to the proposition in [4], we need to show that there is $1 > \alpha > 1/p$ so that the fractional derivative $\langle \xi, \nabla \rangle^\alpha m$ has bounded L^p -multiplier norm independent of ℓ . Our basic idea will be to estimate the fractional derivatives of m of order z with $\operatorname{Re} z = 0$ as L^1 -multipliers, and with $\operatorname{Re} z = \frac{N-1}{2}$ as L^2 -multipliers, followed by applying Stein's interpolation theorem. It turns out that $\frac{N-1}{2}$ is the best possible upper bound on $\operatorname{Re} z$ for that estimate, and that we can also establish (4.3) and (4.4) while bounding these fractional derivatives. However, we encounter some technical difficulties like in [7]. To deal with these, we introduce the Riesz fractional derivative of a function $f: [0, 2] \rightarrow \mathbb{C}$, defined as

$$I^{-z} f(t) = \frac{-1}{\Gamma(-z)} \int_t^2 (u-t)^{-z-1} f(u) \, du,$$

for $\operatorname{Re} z < 0$ and $0 < t \leq 2$. This operator can be extended analytically to the complex plane. For any $k \in \mathbb{N}_{>0}$, assume that f as above is k times differentiable and let $\operatorname{Re} z < k$. Then

$$I^{-z} f(t) = E_{k,f}(z, t) + (-1)^k \frac{1}{\Gamma(k-z)} \int_t^2 (u-t)^{-z+k-1} f^{(k)}(u) \, du, \quad (4.7)$$

where

$$E_{k,f}(z, t) = \sum_{j=0}^{k-1} (-1)^j \frac{(2-t)^{-z+j} f^{(j)}(2)}{\Gamma(j+1-z)}. \quad (4.8)$$

It follows that

$$I^{-k}f(t) = (-1)^k f^{(k)}(t)$$

if f is k times differentiable. Now consider the holomorphic family of multipliers $(m_z)_{z \in \mathbb{C}}$ defined by

$$m_z(\xi) := I^{-z}m(t\xi)|_{t=1}. \quad (4.9)$$

From Müller's work (see also Lemma 7.3 in [5]), it follows that for every $0 < \alpha < 1$,

$$m_\alpha(\xi) - (\langle \xi, \nabla \rangle)^\alpha m(\xi)$$

is an L^q -multiplier for $1 \leq q \leq \infty$, bounded by $\frac{|S|}{\Gamma(1-\alpha)}$. Thus we only need to bound the m_z , and from (4.7) and (4.8), it follows that we need to bound the usual derivatives $\frac{d^k}{dr^k}m(r\xi)$ for $1 \leq r \leq 2$. We have

$$m(\xi) = \prod_{j=1}^{\ell} \tilde{m}(\zeta_j),$$

writing again $\xi = (\zeta_1, \dots, \zeta_\ell)$ with each $\zeta_j \in \mathbb{R}^N$, and

$$\tilde{m}(\zeta) := \widehat{\sigma_{S_R^{N-1}}(\zeta)}.$$

Hence the general Leibniz formula implies that we also need to bound the derivatives of \tilde{m} . We do the latter in the following lemma.

Lemma 4.3. *For each $\alpha \in \mathbb{N}_{>0}$, $r \geq 0$, and $\zeta \in \mathbb{R}^N$, we have*

$$\left(\frac{d}{dr}\right)^\alpha \tilde{m}(r\zeta) = 2\pi^{\alpha+1} r^{-\frac{N-2}{2}} R^{\frac{N}{2}+\alpha} |\zeta|^{-\frac{N-2}{2}+\alpha} \sum_{k=0}^{\alpha} |\pi r R \zeta|^{-k} \frac{\Gamma(\frac{N-2}{2}+k)}{\Gamma(\frac{N-2}{2})} \binom{\alpha}{k} B_{\alpha,k,j}(r, \zeta), \quad (4.10)$$

with

$$B_{\alpha,k,j}(r, \zeta) = \sum_{j=0}^{\alpha-k} (-1)^{j+k} \binom{\alpha-k}{j} J_{\frac{N-2}{2}-\alpha+k+2j}(2\pi r R |\zeta|).$$

Furthermore, there is a constant $C_{\alpha,N}$ such that for every $0 \leq r \leq 2$ and $\zeta \in \mathbb{R}^N$, we have

$$(1 + |\zeta|)^{\frac{N-1}{2}-\alpha} \left| \left(\frac{d}{dr}\right)^\alpha \tilde{m}(r\zeta) \right| \leq C_{\alpha,N}. \quad (4.11)$$

For r and $|\zeta|$ sufficiently small, we also get

$$\left| \left(\frac{d}{dr}\right)^\alpha \tilde{m}(r\zeta) \right| \leq \tilde{C}_{\alpha,N} |\zeta|^{2\alpha}. \quad (4.12)$$

Proof. We have

$$\tilde{m}(\zeta) = 2\pi R^{\frac{N}{2}} |\zeta|^{-\frac{N-2}{2}} J_{\frac{N-2}{2}}(2\pi R|\zeta|),$$

thus we obtain

$$\left(\frac{d}{dr}\right)^\alpha \tilde{m}(r\zeta) = (2\pi)^{\frac{N}{2}} R^{N-1+\alpha} |\zeta|^\alpha \left(\frac{d}{dt}\right)^\alpha [(2\pi t)^{-\frac{N-2}{2}} J_{\frac{N-2}{2}}(2\pi t)] \Big|_{t=r|\zeta|}. \quad (4.13)$$

It is well-known that for every $\nu \in \mathbb{R}$, we have

$$\frac{d}{dt} J_\nu(t) = \frac{1}{2}(J_{\nu-1}(t) - J_{\nu+1}(t)),$$

which extends to

$$\left(\frac{d}{dt}\right)^\alpha J_\nu(t) = \frac{1}{2^\alpha} \sum_{j=0}^\alpha (-1)^j \binom{\alpha}{j} J_{\nu-\alpha+2j}(t).$$

By the Leibniz formula, we thus get

$$\begin{aligned} \left(\frac{d}{dt}\right)^\alpha [t^{-\nu} J_\nu(t)] &= \sum_{k=0}^\alpha (-1)^k \binom{\alpha}{k} \frac{\Gamma(\nu+k)}{\Gamma(\nu)} t^{-\nu-k} J_\nu^{(\alpha-k)}(t) \\ &= \sum_{k=0}^\alpha \sum_{j=0}^{\alpha-k} 2^{-\alpha+k} (-1)^{k+j} \binom{\alpha}{k} \frac{\Gamma(\nu+k)}{\Gamma(\nu)} t^{-\nu-k} \binom{\alpha-k}{j} J_{\nu-\alpha+k+2j}(t), \end{aligned}$$

which we can insert into (4.13). For $\nu = \frac{N-2}{2}$, this adds indeed up to (4.10). Since R depends only on N , all the parameters in (4.10) depend only on N and α . Since for every half-integer ν , $J_\nu(t)$ is $\mathcal{O}(t^{-1/2})$ as $t \rightarrow \infty$, one can see that $\frac{d^\alpha}{dr^\alpha} \tilde{m}(r|\zeta|)$ is $\mathcal{O}(|\zeta|^{\frac{N-1}{2}+\alpha})$ for $|\zeta| \rightarrow \infty$ and each fixed $r \in [1, 2]$. By the series expansion for Bessel functions (2.10), we get

$$\begin{aligned} \tilde{m}(\zeta) &= 2\pi R^{\frac{N}{2}} \frac{|\zeta|^{-\frac{N-2}{2}}}{\Gamma(1/2)} (\pi R|\zeta|)^{\frac{N-2}{2}} \sum_{k=0}^\infty (-1)^k \frac{\Gamma(k+1/2)}{\Gamma(k+N/2)} \frac{(2\pi R|\zeta|)^{2k}}{(2k)!} \\ &= \sum_{k=0}^\infty (-1)^k \frac{(2\pi R|\zeta|)^{2k}}{(2k)!} \prod_{j=0}^{k-1} \frac{j+1/2}{j+N/2}, \end{aligned} \quad (4.14)$$

giving us also

$$\left(\frac{d}{dr}\right)^\alpha \tilde{m}(r\zeta) = (2\pi R|\zeta|)^{2\lfloor \frac{\alpha+1}{2} \rfloor} \sum_{2k \geq \alpha} (-1)^k r^{2k-\alpha} \frac{(2\pi R|\zeta|)^{2(k-\lfloor \frac{\alpha+1}{2} \rfloor)}}{(2k)!} \prod_{j=0}^{k-1} \frac{j+1/2}{j+N/2}.$$

This implies boundedness of $\frac{d^\alpha}{dr^\alpha} \tilde{m}(r|\zeta|)$ for $|\zeta|$ close to 0, leading to both (4.11) and (4.12). By now, the bounds might depend on r , but if we insert $r = 0$ and $r = 2$, we can conclude the lemma by continuity. **q.e.d.**

With this, we can estimate the derivatives of m straight forward.

Lemma 4.4. For every $\xi \in \mathbb{R}^n$, every $0 \leq r \leq 2$ and every integer j with $0 \leq j \leq \lceil \frac{N-1}{2} \rceil$, we have

$$|\xi|^{\frac{N-1}{2}-j} \left| \left(\frac{d}{dr} \right)^j m(r\xi) \right| \leq C_N. \quad (4.15)$$

Remark. We have the restriction $j \leq \lceil \frac{N-1}{2} \rceil$ only because we want to have C_N instead of $C_{j,N}$. For bigger j , the argument still holds.

Proof. Pick a fixed constant A_N so that for every $j \leq \lceil \frac{N-1}{2} \rceil$ and $0 \leq r \leq 2$, we have

$$\left| \left(\frac{d}{dr} \right)^j m(r\zeta) \right| \leq (A_N(1 + |\zeta|))^{-\frac{N-1}{2}+\alpha}$$

uniformly. Now the general Leibniz formula gives us

$$\begin{aligned} \left| \left(\frac{d}{dr} \right)^j m(r\xi) \right| &= \left| \sum_{\substack{\alpha \in \mathbb{N}^\ell \\ |\alpha|=j}} \binom{j}{\alpha} \prod_{k=1}^\ell \left(\frac{d}{dr} \right)^{\alpha_k} \tilde{m}(r\zeta_k) \right| \\ &\leq \sum_{|\alpha|=j} \binom{j}{\alpha} \left| \prod_{A_N|\zeta_k| \geq 2} \left(\frac{d}{dr} \right)^{\alpha_k} \tilde{m}(r\zeta_k) \right| \cdot \left| \prod_{A_N|\zeta_k| < 2} \left(\frac{d}{dr} \right)^{\alpha_k} \tilde{m}(r\zeta_k) \right| \end{aligned}$$

Put

$$D^2 = \sum_{A_N|\zeta_k| \geq 2} |\zeta_k|^2, \quad E^2 = \sum_{A_N|\zeta_k| < 2} |\zeta_k|^2.$$

Assume $D > 0$, i.e. there is k with $A_N|\zeta_k| \geq 2$. Otherwise, (4.15) follows by the same argument. By (4.14), there is a constant b_N such that if $A_N|\zeta_k| < 2$, we have

$$|\tilde{m}(\zeta_k)| \leq e^{-b_N|\zeta_k|^2} = e^{-b_N|\zeta_k|^2} (C_N|\zeta_k|^2)^0.$$

With that, (4.12) implies

$$\left| \left(\frac{d}{dr} \right)^{\alpha_k} \tilde{m}(r\zeta_k) \right| = e^{-b_N|\zeta_k|^2} \left| e^{b_N|\zeta_k|^2} \left(\frac{d}{dr} \right)^{\alpha_k} \tilde{m}(r\zeta_k) \right| \leq e^{-b_N|\zeta_k|^2} (C_N|\zeta_k|^2)^{\alpha_k}$$

if $\alpha_k > 0$ and $A_N|\zeta_k| < 2$. Together with (4.11), this implies

$$\begin{aligned} &|\xi|^{\frac{N-1}{2}-j} \cdot \left| \left(\frac{d}{dr} \right)^j m(r\xi) \right| \\ &\leq |\xi|^{\frac{N-1}{2}-j} \sum_{|\alpha|=j} \binom{j}{\alpha} \left[\overbrace{\prod_{A_N|\zeta_k| \geq 2} (A_N(1 + |\zeta_k|))^{-\frac{N-1}{2}+\alpha_k}}^{=: \Pi_1} \right] \cdot \left[e^{-b_N E^2} \overbrace{\prod_{A_N|\zeta_k| < 2} (C_N|\zeta_k|^2)^{\alpha_k}}^{=: \Pi_2} \right] \\ &= |\xi|^{-j} \sum_{|\alpha|=j} \binom{j}{\alpha} \left[(A_N D)^{\frac{N-1}{2} \Pi_1} \right] \cdot \left[\frac{|\xi|^{\frac{N-1}{2}}}{(A_N D)^{\frac{N-1}{2}}} e^{-b_N E^2 \Pi_2} \right], \end{aligned} \quad (4.16)$$

Since $A_N D \geq 2$, we obviously have

$$(A_N D)^2 \leq \prod_{A_N |\zeta_k| \geq 2} (A_N |\zeta_k|)^2.$$

Hence

$$(A_N D)^{\frac{N-1}{2}} \prod_{A_N |\zeta_k| \geq 2} (A_N (1 + |\zeta_k|))^{-\frac{N-1}{2} + \alpha_k} \leq \prod_{A_N |\zeta_k| \geq 2} (A_N |\zeta_k|)^{\alpha_k} \quad (4.17)$$

for each α . Furthermore, we have

$$|\xi|^{\frac{N-1}{2}} \leq (D + E)^{\frac{N-1}{2}} \leq 2^{\frac{N-1}{2}} (D^{\frac{N-1}{2}} + E^{\frac{N-1}{2}}),$$

giving us

$$\frac{|\xi|^{\frac{N-1}{2}}}{(A_N D)^{\frac{N-1}{2}}} e^{-b_N E^2} \leq \left(\frac{1}{A_N^{\frac{N-1}{2}}} + \frac{E^{\frac{N-1}{2}}}{(A_N D)^{\frac{N-1}{2}}} \right) e^{-b_N E^2} \leq \left(\frac{1}{A_N^{\frac{N-1}{2}}} + E^{\frac{N-1}{2}} \right) e^{-b_N E^2} \leq C_N. \quad (4.18)$$

Also, if $A_N |\zeta_k| < 2$, we have

$$|\zeta_k|^2 < \frac{2}{A_N} |\zeta_k| \leq C_N |\zeta_k|. \quad (4.19)$$

Altogether, (4.17), (4.18), and (4.19) imply

$$\begin{aligned} (4.16) &\leq C_N |\xi|^{-j} \sum_{|\alpha|=j} \binom{j}{\alpha} \left[\prod_{A_N |\zeta_k| \geq 2} (A_N |\zeta_k|)^{\alpha_k} \right] \cdot \left[\prod_{A_N |\zeta_k| < 2} (C_N |\zeta_k|)^{\alpha_k} \right] \\ &\leq C_N |\xi|^{-j} \sum_{|\alpha|=j} \binom{j}{\alpha} \prod_{k=1}^{\ell} (C_N |\zeta_k|)^{\alpha_k} \\ &= C_N |\xi|^{-j} (C_N |\xi|)^j < C_N, \end{aligned}$$

finishing the estimate. **q.e.d.**

Now, we are able to show that $\sigma_S - P \, dx$ fulfills (4.3) and (4.3), which allows us to apply Carbery's interpolation argument.

Lemma 4.5. *There is a constant C_N such that for every $\theta \in S^{n-1}$ and every $u \in \mathbb{R}^*$*

$$\begin{aligned} |m(u\theta) - \hat{P}(u\theta)| &\leq C_N \cdot \min\{|u|, |u|^{-1}\}, \\ |\langle \theta, \nabla(m - \hat{P})(u\theta) \rangle| &\leq C_N \cdot \min\{1, |u|^{-1}\}. \end{aligned}$$

Proof. We clearly have

$$|u \cdot \hat{P}(u\theta)| \leq C, \quad |1 - \hat{P}(u\theta)| \leq C|u|, \quad \text{and} \quad |\langle \theta, \nabla \hat{P}(u\theta) \rangle| \leq C \cdot \min\{1, |u|^{-1}\}.$$

From Lemma 4.3, we also know that

$$|u \cdot m(u\theta)| \leq C_N \quad \text{and} \quad |\langle \theta, \nabla m(u\theta) \rangle| \leq C_N \cdot \min\{1, |u|^{-1}\},$$

since both $\frac{d}{du}m(u\theta)$ and $\langle u\theta, \nabla m(u\theta) \rangle$ are bounded, proving the second inequality. Also, from (4.14), we get that

$$|1 - m(u\theta)| \leq 1 - e^{-c_N u^2} \leq C_N u^2 \leq C_N |u|$$

if $|u| \leq 1$, while $|1 - m(u\theta)| < |u|$ is obvious for $|u| > 1$. Since

$$|m(u\theta) - \hat{P}(u\theta)| \leq |m(u\theta)| + |\hat{P}(u\theta)| \quad \text{and} \quad |m(u\theta) - \hat{P}(u\theta)| \leq |1 - m(u\theta)| + |1 - \hat{P}(u\theta)|,$$

the lemma follows. **q.e.d.**

This leaves us with finding some α with $1/p < \alpha < 1$ such that m_α is an L^p -multiplier. We can't interpolate the corresponding multiplier operators of the family $(m_z)_{0 \leq \operatorname{Re} z \leq \frac{N-1}{2}}$ directly, but interpolation is still possible. For this, assume that $\frac{N}{N-1} < p < \frac{N-1}{N-2}$, and let $1/p = 1 - \theta + \theta/2$, i.e. $\theta = 2 - 2/p$. Then $\frac{N-1}{2}\theta < 1 < \frac{N}{2}\theta$. Fix ε such that $0 < \varepsilon < \frac{N}{2}\theta - 1$, and set

$$\alpha := \frac{N-1}{2}\theta - \varepsilon.$$

Then $\alpha > 1 - \theta/2 = 1/p$. If N is odd, $\frac{N-1}{2}$ is an integer, and one can use the formulas (4.7) and (4.8) to argue as in Section 7.3 and Lemma 7.5 of [5]. We get that m_z is an L^1 -multiplier for $\operatorname{Re} z = -\varepsilon$ and an L^2 -multiplier for $\operatorname{Re} z = \frac{N-1}{2} - \varepsilon$ so that the family

$$(m_{\frac{N-1}{2}z - \varepsilon})_{0 \leq \operatorname{Re} z \leq 1}$$

is analytic and of admissible growth in the sense of Stein's interpolation theorem. For $p > \frac{N-1}{N-2}$, we can interpolate with the endpoint ∞ . This already suffices to prove Theorem 1, since one could simply go from N to $N+1$ if N is even, but for even N , Theorem 2' still holds. In that case, we need to interpolate twice, paying further attention to the upcoming bounds of the multiplier operators. Suppose we have an analytic family of operators (T_z) with $0 \leq \operatorname{Re} z \leq 1$ so that

$$\|T_{it}\|_{p_0 \rightarrow p_0} \leq M_0(t) \quad \text{and} \quad \|T_{1+it}\|_{p_1 \rightarrow p_1} \leq M_1(t),$$

and that we have $b < \pi$ such that

$$\sup_{t \in \mathbb{R}} e^{-b|t|} \log M_j(t) < \infty,$$

$j = 0, 1$. Then $\|T_\theta\|_{p \rightarrow p} \leq M(\theta)$ for $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and for instance from [6, p. 38], it follows that

$$M(t) = \exp \left[\frac{\sin(\pi t)}{2} \int_{\mathbb{R}} \left(\frac{\log M_0(y)}{\cosh(\pi y) - \cos \pi t} + \frac{\log M_1(y)}{\cosh(\pi y) + \cos \pi t} \right) dy \right]. \quad (4.20)$$

For $0 < \varepsilon < 1$, we consider the multipliers $m_z^\varepsilon(\xi) = (1 + |\xi|)^{\frac{N-1}{2} - \varepsilon - z} m_z(\xi)$, similarly as in [7]. For $-\varepsilon \leq \operatorname{Re} z \leq \frac{N}{2}$, Lemma 7.4 in [5] gives us the precise bounds

$$|m_{r+it}^\varepsilon(\xi)| \leq C_N \varepsilon^{-1} (1 + t^2)^{\frac{N+1}{4}} e^{\frac{\pi t}{2}}.$$

Hence, for every $t \in \mathbb{R}$, the family $(m_{\frac{N}{2}z-\varepsilon+it}^\varepsilon)_z$ of L^2 -multipliers has admissible growth for $0 \leq \operatorname{Re} z \leq 1$, and by interpolation, (4.20) implies

$$|m_{\frac{N-1}{2}-\varepsilon+it}(\xi)| \leq e^{C_N|t|}.$$

Thus we also have admissible growth for $\operatorname{Re} z = \frac{N-1}{2} - \varepsilon$, N even, and taking ε and α as above, this proves Theorem 2'.

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